S.-T. Yau College Student Mathematics Contest

Applied and Computational Mathematics 2015 (Individual)

1. Suppose an n by n matrix A is given by

$$A = \begin{pmatrix} 1 & r & & & & \\ & 1 & r & & & \\ & & 1 & r & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 1 & r \\ r & & & & & 1 \end{pmatrix}_{n\times n}$$

$A\mathbf{x} = \mathbf{b}$, prove that

$$\|\mathbf{x}\| \le C\|\mathbf{b}\|,$$

where the constant C is independent of the dimension n.

2. For an interval $[a, b]$, we divide it into $N + 1$ equally spaced subintervals by using the nodal points:

$$a = x_0 < x_1 < \cdots < x_{N+1} = b,$$

with

$$x_i = a + i\,h, \quad h = (b - a)/(N + 1).$$

For any continuous function w on $[0, \pi]$, we define $\Pi_h w$ to be the piecewise linear interpolation of w, namely $\Pi_h w$ is linear on each subinterval (x_i, x_{i+1}) for $i = 0, 1, \cdots, N$, and it takes the same values as w at all nodal points x_i, $i = 0, 1, \cdots, N + 1$. For any function w, we define

$$\|w\| = \left(\int_0^\pi w^2(x)dx\right)^{1/2}.$$

Prove the following estimates for any function $u \in C^2[0, \pi]$:

$$\|u - \Pi_h u\| \le \frac{1}{\pi^2}h^2\,\|u''\|\,, \quad \|u' - (\Pi_h u)'\| \le \frac{1}{\pi}h\,\|u''\|\,.$$

3. Newton iteration for computing the kth root ($k \ge 2$) of $C > 0$ is

$$x_{n+1} = x_n - \frac{x_n^k - C}{kx_n^{k-1}}.$$

Show that the iteration converges for any initial value $x_0 > 0$.